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Finite elastic eversion of a thick-walled incompressible spherical cap

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Abstract. For perfectly elastic rubber-like materials, which are capable of undergoing extremely large deformations, the number of exact solutions remains limited, especially in the context of fully three-dimensional deformations. Here a simple exact solution describing the finite elastic eversion of a sector of a thick-walled incompressible spherical shell is determined for the modified Varga elastic material. This new solution, which describes a portion of a spherical shell being turned inside out, is deduced from a known simplified system and it is shown, by solving the full equilibrium equations, that no further solutions of this type can be deduced for this particular material. Further, a general family of response functions is considered, which involves an arbitrary index n, and which incorporates standard materials such as the neo-Hookean and Varga strain-energy functions. It is established that other than $n = 1$ (namely the Varga material) only the special case $n = 2$ admits nontrivial solutions to the eversion problem, but the resulting second-order highly nonlinear ordinary differential equation appears not to admit any simple analytical solutions. Finally, the new solution is examined as a potential solution of the 'snap-buckling' problem of a spherical cap. Unfortunately, the solution appears not to be applicable to this problem and instead it is presented in the specific context of the eversion of a thick-walled spherical cap, with no applied forces acting on one of the surfaces of the deformed configuration.

Key words: perfectly elastic, incompressible, Varga strain-energy, spherical eversion, exact solution.

1. Introduction

For isotropic incompressible perfectly elastic materials, the governing equations for static finite deformations are highly nonlinear, and, apart from the controllable deformations, there exist only a limited number of known exact deformations, which apply to various restricted forms of the strain-energy function. Moreover, the determination of fully three-dimensional deformations, in contrast to plane strain deformations, presents additional difficulties. A number of exact axially symmetric and fully three-dimensional deformations are given in Hill [1, 2] for a neo-Hookean elastic material, while Hill and Arrigo [3] and Arrigo and Hill [4] present a number of new integrals and new exact solutions for axially symmetric deformations of the Varga and modified Varga strain-energy functions. The results given in [3, 4] are utilized in Hill and Arrigo [5] to solve the problem of the stability of a thick-walled spherical shell which is subjected to external pressure. In this paper we also exploit the results given in [3, 4] to determine a simple exact deformation applying to the problem of eversion of a thick-walled spherical cap.

The neo-Hookean and Varga elastic materials have strain-energy functions, respectively, given by

$$\Sigma = \frac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \qquad \Sigma = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3), \tag{1.1}$$

where λ_i $(i = 1, 2, 3)$ denote the principal stretches such that $\lambda_1\lambda_2\lambda_3 = 1$ and μ in both cases is the usual linear elastic shear modulus. Both strain-energy functions are known to apply over restricted ranges of deformation, but the neo-Hookean material is generally regarded as a better model than the Varga material over a variety of deformations. The new exact solution presented here applies to the so-called modified Varga material, which was first introduced in [3], and has the strain-energy function

$$\Sigma = \alpha(\lambda_1 + \lambda_2 + \lambda_3 - 3) + \beta\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - 3\right), \tag{1.2}$$

where α and β are material constants such that $\alpha + \beta = 2\mu$. As shown in [5], the additional material constant greatly improves the range of physical applicability over the single-term Varga strain-energy function $(1.1)_2$.

The problem of the 'snap-buckling' of a portion of a spherical cap is a familiar one and here we examine the new solution as a potential solution of this problem. The 'snap-buckling' problem is particularly interesting, because the body can remain in a deformed state with no applied forces acting. Even if we replace the requirement of point-wise vanishing of the stress, by the requirement that the average forces are zero, we are unfortunately, unable to determine the arbitrary constants in the new solution such that this is the case. Accordingly, we examine the greatly simplified problem, where only one of spherical surfaces is stress-free. It is at least clear from this problem, that the case of the two spherical surfaces being stress-free is not embodied in the new solution.

In this paper we consider a deformation describing a portion of a spherical shell being turned inside out. Namely, we consider the axially symmetric eversion of a sector of a thick-walled spherical shell, and in material and spatial spherical polar coordinates (R, Θ, Φ) and (r, θ, ϕ), respectively, we examine a deformation of the form

$$r = (-R^3 + f(\Theta))^{1/3}, \qquad \theta = \pi - \Theta, \qquad \phi = \Phi, \tag{1.3}$$

where $f(\Theta)$ is a function of Θ only. In the following section we show that, although there are no non-trivial $f(\Theta)$ for the neo-Hookean material, other than the controllable deformation $f(\Theta) =$ constant, due to the Green and Shield [6], the modified Varga material with strain-energy function (1.2) admits the simple solution

$$f(\Theta) = k(\cos 2\Theta)^{-3/2}, \tag{1.4}$$

where k is a constant. This elegant simple result is derived by means of the integrals given in [3, 4] and one objective of this work is to investigate the extent to which this solution can be extended. In particular, we investigate if the modified Varga material admits a more general exact solution perhaps involving further arbitrary constants. We also examine (1.3) for a family of response functions (see Equation (4.1)) which are characterised by an index n, which includes the neo-Hookean and Varga materials for the cases $n = 0$ and $n = 1$, respectively. We are able to show that only the case $n = 2$ gives rise to nontrivial $f(\Theta)$, except that in this case we are unable to integrate fully the resulting highly nonlinear ordinary differential equation (see Equations (5.4) and (5.5)).

We emphasise that, in order to obtain tractable equations, we need to make certain assumptions regarding the response coefficients ϕ_1 and ϕ_2 which are defined by (2.5). The single assumption (4.1) is the simplest which incorporates the standard materials (1.1) and yet enjoys certain analytical advantages for this particular problem. However, we note that in general there appears to be no obvious 'simple' properly invariant $\Sigma(\lambda_1, \lambda_2, \lambda_3)$ which gives rise to either of the expressions (4.2). But this does not necessarily mean that such strain-energy functions do not exist, and we could, if necessary, deduce a properly invariant $\Sigma(I_1, I_2)$ from (2.5) and (2.8) and using (2.6) to express I and λ as functions of I_1 and I_2, which are defined by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}. \tag{1.5}$$

The resulting expression would, however, involve the roots of a cubic and we do not investigate such details here.

In the following section we state the basic governing equations for axially symmetric deformations of perfectly elastic materials. In addition, we establish there that the neo-Hookean material does not admit any solutions for $f(\Theta)$ other than $f(\Theta) =$ constant and we also derive (1.4) for the modified Varga strain-energy function. In Section 3 we show from the general equations that (1.4) is the most general solution applying to the modified Varga material. In Section 4 we suppose that the elastic material has response function ϕ_1 given by (4.1) and we show that only the cases $n = 1$ and $n = 2$ give rise to nontrivial solutions for $f(\Theta)$ and the latter case is described separately in the subsequent section. In the final section of the paper we apply the solution (1.4) to the problem of the eversion of a thick-walled spherical cap and certain complicated integrals for resultant forces are evaluated in the Appendices.

2. General equations for axially symmetric deformations

For isotropic incompressible hyperelastic materials, the axially symmetric deformation in spherical polar coordinates,

$$r = r(R, \Theta), \qquad \theta = \theta(R, \Theta), \qquad \phi = \Phi, \tag{2.1}$$

satisfies the incompressibility condition

$$r_R\theta_\Theta - r_\Theta\theta_R = \frac{R^2 \sin\Theta}{r^2 \sin\theta}, \tag{2.2}$$

where, as usual, subscripts denote partial derivatives. The equilibrium equations can be shown to become

$$\begin{aligned} p_r &= \phi_1 \left\{ \nabla^2 r - r\left(\theta_R^2 + \frac{\theta_\Theta^2}{R^2}\right)\right\} + \phi_{1R} r_R + \frac{\phi_{1\Theta} r_\Theta}{R^2} - \phi_2 \frac{r \sin^2\theta}{R^2 \sin^2\Theta}, \\ \frac{p_\theta}{r^2} &= \phi_1 \left\{ \nabla^2 \theta + \frac{2}{r}\left(r_R\theta_R + \frac{r_\Theta\theta_\Theta}{R^2}\right)\right\} + \phi_{1R}\theta_R + \frac{\phi_{1\Theta}\theta_\Theta}{R^2} - \phi_2 \frac{\sin\theta\cos\theta}{R^2 \sin^2\Theta}, \end{aligned} \tag{2.3}$$

where p is the pressure function and ∇^2 is the Laplacian given by

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2} + \frac{\cot\Theta}{R^2}\frac{\partial}{\partial\Theta} \tag{2.4}$$

and the response functions ϕ_1 and ϕ_2 are given by

$$\phi_1 = 2\left\{\frac{\partial\Sigma}{\partial I_1} + \lambda^2\frac{\partial\Sigma}{\partial I_2}\right\}, \qquad \phi_2 = 2\left\{\frac{\partial\Sigma}{\partial I_1} + \left(I - \frac{1}{\lambda^4}\right)\frac{\partial\Sigma}{\partial I_2}\right\}, \tag{2.5}$$

where $\Sigma(I_1, I_2)$ is the strain-energy function of the material. Further, I_1 and I_2 are the first two invariants of the inverse Cauchy deformation tensor which are given by

$$I_1 = I + \lambda^2, \qquad I_2 = \lambda^2 I + \frac{1}{\lambda^2}, \tag{2.6}$$

where I and λ are defined by

$$I = r_R^2 + \frac{r_\Theta^2}{R^2} + r^2\left(\theta_R^2 + \frac{\theta_\Theta^2}{R^2}\right), \qquad \lambda = \frac{r\sin\theta}{R\sin\Theta}, \tag{2.7}$$

and in terms of I and λ, the response functions ϕ_1 and ϕ_2 as given by (2.5) become

$$\phi_1 = 2\frac{\partial\Sigma}{\partial I}, \qquad \phi_2 = 2\frac{\partial\Sigma}{\partial\lambda^2}. \tag{2.8}$$

We note that for prescribed $\phi_1(I, \lambda)$ and $\phi_2(I, \lambda)$ we can determine the partial derivatives $\partial\Sigma/\partial I_1$ and $\partial\Sigma/\partial I_2$ from the relations (2.5) provided the Jacobian of (2.6) is finite.

For the neo-Hookean material $\phi_1 = \phi_2 = \mu$ and it is simplest to show directly from (2.3) that no non-trivial solutions exist for $f(\Theta)$. Assuming (1.3), we have

$$r_R = -\left(\frac{R}{r}\right)^2, \qquad r_\Theta = \frac{f'}{3r^2}, \tag{2.9}$$

where primes here and throughout denote differentation with respect to Θ, and it is a simple matter to show that the incompressibility condition (2.2) is satisfied. For the neo-Hookean material we have from $(2.3)_2$

$$p_\theta = -\frac{2\mu}{3}\frac{f'}{rR^2} = \frac{2\mu}{3}\frac{\frac{\mathrm{d}f}{\mathrm{d}\theta}}{r(-r^3 + f)^{2/3}},$$

which on integration yields

$$p(r, \theta) = 2\mu\frac{R}{r} + \mu F^*(r), \tag{2.10}$$

where $F^*(r)$ denotes an arbitrary function of r, which in principle, is determined by substitution of (2.10) in the first equilibrium equation $(2.3)_1$, thus

$$r^2(-r^3 + f)^{2/3}\frac{\mathrm{d}F^*}{\mathrm{d}r}(r) = \tfrac{1}{3}\{f'' + \cot\Theta f' + 6f\} - \frac{2}{r^3}\left(f^2 + \frac{f'^2}{9}\right). \tag{2.11}$$

Only in the case $f(\Theta) = \text{constant}$ does (2.11) reduce to a well-defined equation for the determination of $F^*(r)$.

In [3, 4, 5] it is shown that the integral

$$(Rr_R + r\theta_\Theta)\sin(\theta - \Theta) + (rR\theta_R - r_\Theta)\cos(\theta - \Theta) = 0, \tag{2.12}$$

together with (2.2), constitutes a second-order system of equations, every solution (r, θ) of which is a bona-fide solution of the fourth-order system defined by (2.2) and (2.3) for the modified Varga elastic material which has a strain-energy function defined by (1.2). Moreover, any solution of (2.2) and (2.12) has a pressure function given explicitly by

$$p = -\frac{(\phi_1 + \beta)}{\lambda} + p_0, \tag{2.13}$$

where p_0 is a constant and ϕ_1 and λ are as defined above. From (1.3) and (2.12) it is a simple matter to deduce

$$f' = 3f\tan 2\Theta, \tag{2.14}$$

from which we may readily obtain the exact solution (1.4). In the following section we show directly from (2.3) that this is the only nonconstant solution applying to the modified Varga material.

3. Modified Varga strain-energy function

Given the derivation of (1.4) from the integral (2.12), it is natural to ask whether the full equilibrium equations for the modified Varga elastic material admit a more general $f(\Theta)$. We are able to demonstrate that this in fact is not the case and that there exist only two solutions, namely (1.4) and $f(\Theta) = \text{constant}$. From (1.3) and (2.7) we can deduce

$$I = -\frac{2}{\lambda} + \frac{A(\Theta)}{r^4R^2}, \quad \lambda = \frac{r}{R}, \tag{3.1}$$

where the function $A(\Theta)$ is defined by

$$A(\Theta) = f^2 + \frac{f'^2}{9}. \tag{3.2}$$

Now from the relations $I = \lambda_1^2 + \lambda_2^2,\ \ \lambda_3 = \lambda$, we find that (1.2) becomes

$$\Sigma = \alpha\left(\left(I + \frac{2}{\lambda}\right)^{1/2} + \lambda - 3\right) + \beta\left(\lambda\left(I + \frac{2}{\lambda}\right)^{1/2} + \frac{1}{\lambda} - 3\right), \tag{3.3}$$

and therefore from (2.8) we find that the response functions ϕ_1 and ϕ_2 are given by

$$\phi_1 = \frac{(\alpha + \beta\lambda)}{(I + \frac{2}{\lambda})^{1/2}}, \qquad \phi_2 = -\frac{\phi_1}{\lambda^3} + \frac{\alpha}{\lambda} + \frac{\beta}{\lambda}\left(I + \frac{2}{\lambda}\right)^{1/2} - \frac{\beta}{\lambda^3}. \tag{3.4}$$

On introducing $B(\Theta) = A(\Theta)^{-1/2}$ we may deduce from the above relations

$$\left(I + \frac{2}{\lambda}\right)^{1/2} = \frac{1}{B(\Theta)r^2R}, \qquad \phi_1 = B(\Theta)r^2(\alpha R + \beta r). \tag{3.5}$$

From (1.3) and using the above expressions for the response functions ϕ_1 and ϕ_2 in the equilibrium equations (2.3) and the relations

$$p_R = -p_r \left(\frac{R}{r}\right)^2, \qquad p_\Theta = -p_\theta + p_r \frac{f'}{3r^2}, \tag{3.6}$$

we can after an extremely tedious calculation show that the equilibrium equations reduce to

$$\begin{aligned} q_R &= -(\alpha R + \beta r)\frac{F(\Theta)}{r^2}, \\ q_\Theta &= (\alpha R + \beta r)\frac{f' F(\Theta)}{3r^2 R^2} + (\alpha R + \beta r)\frac{rG(\Theta)}{R^2} + \beta H(\Theta), \end{aligned} \tag{3.7}$$

where $F(\Theta)$, $G(\Theta)$ and $H(\Theta)$ are defined by

$$\begin{aligned} F(\Theta) &= \frac{(Bf')'}{3} + \cot\Theta \frac{Bf'}{3} - 3Bf - 1, \\ G(\Theta) &= (Bf)' + \cot\Theta Bf + Bf' - \cot\Theta, \\ H(\Theta) &= \frac{f'}{3} + \cot\Theta f - \frac{\cot\Theta}{B}, \end{aligned} \tag{3.8}$$

and, where motivated by (2.13), we have introduced q defined by

$$q = p + \frac{(\phi_1 + \beta)}{\lambda}. \tag{3.9}$$

From (1.4) and (3.8), it is a simple matter to show that F, G and H are identically zero, which confirms (2.13). The question is, however, whether (3.7) admits a more general $f(\Theta)$. On equating expressions for $q_{R\Theta}$ we may deduce from (3.7)

$$(\alpha R + \beta r)F' = \frac{(\alpha R + 2\beta r)}{R^3}\left(\frac{Ff'}{3} + Gf\right), \tag{3.10}$$

which holds only if

$$F' = 0, \quad \frac{Ff'}{3} + Gf = 0. \tag{3.11}$$

Now from the definition of $B(\Theta)$ we have

$$B^2 = \left(f^2 + \frac{f'^2}{9}\right)^{-1}, \tag{3.12}$$

and therefore

$$B\left(f^2 + \frac{f'^2}{9}\right) = \frac{1}{B}. \tag{3.13}$$

On expanding the left-hand side of (3.14) and using (3.13) it is not difficult for us to show from (3.8) that the following identity holds

$$\frac{Ff'}{3} + Gf = -H, \tag{3.14}$$

and therefore we require $F' = 0$ and $H = 0$. From these two conditions we may deduce

$$\frac{(Bf')'}{3} + \frac{Bf'}{3}\cot\Theta - 3Bf = \gamma, \qquad \frac{Bf'}{3} = \cot\Theta - Bf\cot\Theta, \tag{3.15}$$

where γ is an arbitrary constant. By eliminating Bf' from these equations we obtain the linear first-order ordinary differential equation for Bf, thus

$$\cot\Theta(Bf)' + 2(Bf) = -(\gamma + 1). \tag{3.16}$$

On integration and redesignating the two arbitrary constants to C_1 and C_2, we may eventually deduce

$$Bf = C_1\cos 2\Theta + C_2, \qquad \frac{Bf'}{3} = -\cot\Theta(C_1\cos 2\Theta + C_2 - 1), \tag{3.17}$$

which by squaring and adding using (3.12) simplifies to give

$$(C_1x + C_2 - 1)(C_1x + C_2 - x) = 0, \tag{3.18}$$

where x here denotes $\cos 2\Theta$. Clearly, there are only two possibilities, namely either $C_1 = 0$ and $C_2 = 1$ which corresponds to $f(\Theta) =$ constant, or $C_1 = 1$ and $C_2 = 0$ which corresponds to (1.4). Thus, at least for the modified Varga material, we have established that there are no more general solutions for the deformation (1.3) other than (1.4) and $f(\Theta) =$ constant. In the following section we attempt to find further materials which admit nontrivial deformations of the form (1.3) by prescribing the form of the first response coefficient ϕ_1.

4. Elastic material with response function characterised by an index *n*

In this section we suppose that the first response function ϕ_1 is given by

$$\phi_1 = \frac{2^n\mu}{(I + \frac{2}{\lambda})^{n/2}}, \tag{4.1}$$

where the normalising constant is determined by the condition ϕ_1 tends to μ as I tends to two and λ tends to unity. The form of this expression is motivated by the fact that $I + 2/\lambda$ has an 'almost' separable structure and therefore ϕ_1 also enjoys this feature. In addition (4.1) includes both the neo-Hookean and Varga elastic materials for $n = 0$ and $n = 1$, respectively. From $(2.8)_1$ and (4.1) we may deduce

$$\begin{aligned}\Sigma(I,\lambda) &= \frac{2^n\mu}{(2-n)}\left(I + \frac{2}{\lambda}\right)^{1-n/2} + \Sigma_0(\lambda), \quad n \neq 2,\\ \Sigma(I,\lambda) &= 2\mu\log\left(I + \frac{2}{\lambda}\right) + \Sigma_0(\lambda), \quad n = 2,\end{aligned} \tag{4.2}$$

where in both cases $\Sigma_0(\lambda)$ denotes an arbitrary function of λ. We note that it is not immediately clear from (4.2) as to the precise form of a properly invariant 'simple' strain-energy function $\Sigma(\lambda_1, \lambda_2, \lambda_3)$, which could give rise to such expressions. However, this does not necessarily imply that such a $\Sigma(\lambda_1, \lambda_2, \lambda_3)$ does not exist and as noted in the introduction, we could if necessary deduce such a $\Sigma(I_1, I_2)$ from (2.5) and (2.8) and using the relations (2.6) and (1.5), but the result would be complicated and involve the roots of a cubic. For our purposes Equation (4.1) is a simple assumption, which enjoys certain analytical advantages, and which encompasses two standard materials. From $(2.8)_2$ and (4.2) we may deduce

$$\phi_2 = -\frac{\phi_1}{\lambda^3} + \sigma(\lambda), \tag{4.3}$$

where $\sigma(\lambda) = 2\partial\Sigma_0/\partial\lambda^2$ and which for the time being we leave arbitrary. Now, on introducing $B(\Theta)$ defined by

$$B(\Theta) = A(\Theta)^{-n/2}, \tag{4.4}$$

which coincides with that used in the previous section with $n = 1$, we may deduce from $(3.1)_1$ and (4.1)

$$\phi_1 = 2^n \mu\, B(\Theta)\, r^{2n} R^n. \tag{4.5}$$

From (1.3), (4.3) and (4.5) the equilibrium equations (2.3) eventually simplify to give

$$\begin{aligned} Q_R &= -r^{2n-4} R^n C(\Theta) - 2(n-1) r^{2n-7} R^n B(\Theta)^{1-2/n} + \frac{\bar\sigma(\lambda)}{r}, \\ Q_\Theta &= r^{2n-4} R^{n-2} \frac{f'}{3} C(\Theta) + r^{2n-1} R^{n-2} D(\Theta) \\ &\quad + \frac{2}{3}(n-1) r^{2n-7} R^{n-2} f' B(\Theta)^{1-2/n} - \lambda^2 \bar\sigma(\lambda) \left(\frac{f'}{3r^3} + \cot\Theta\right), \end{aligned} \tag{4.6}$$

where the functions $C(\Theta)$ and $\mathrm{D}(\Theta)$ are defined by

$$\begin{aligned} C(\Theta) &= \frac{(Bf')'}{3} + \cot\Theta \frac{Bf'}{3} - 3nBf, \\ D(\Theta) &= (Bf)' + \cot\Theta Bf + nBf', \end{aligned} \tag{4.7}$$

and we have introduced Q and $\bar\sigma(\lambda)$ such that

$$Q = \frac{1}{2^n \mu}\left(p + \frac{\phi_1}{\lambda}\right), \qquad \bar\sigma(\lambda) = \frac{\sigma(\lambda)}{2^n \mu}. \tag{4.8}$$

We comment that in the derivation of (4.6) we have made use of Equation (3.6) and we exclude the case $n = 0$ from the discussion. We also note that from (3.2), (4.4) and the definition (4.7) of $C(\Theta)$ and $D(\Theta)$ we can establish the important identity

$$C\frac{f'}{3} + Df = B^{-2/n}\left\{\left(1 - \frac{1}{n}\right) B' + \cot\Theta B\right\}. \tag{4.9}$$

On equating expressions for $Q_{R\Theta}$ we may eventually deduce

$$r^{2n-4}\left\{R^n C' + (n-2)R^{n-3}\frac{Cf'}{3}\right\} + 2(n-1)(n-2)r^{2n-7}\left\{R^n\frac{B^{-2/n}}{n}B'\right.$$
$$\left. + R^{n-3}B^{1-2/n}\frac{f'}{3}\right\} + \{(n-2)r^{2n-1}R^{n-3} - (2n-1)r^{2n-4}R^n\}D$$
$$+\frac{1}{r^2R^2}\left\{\frac{f'}{3} + f\cot\Theta\right\}\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^2\bar{\sigma}(\lambda)) = 0. \tag{4.10}$$

Clearly, $n = 1$ and $n = 2$ constitute special cases and the case $n = 2$ is considered in detail in the following section. For $n = 1$ we have for the Varga elastic material

$$\Sigma_0(\lambda) = 2\mu(\lambda - 3), \qquad \sigma(\lambda) = \frac{2\mu}{\lambda}, \qquad \bar{\sigma}(\lambda) = \frac{1}{\lambda}, \tag{4.11}$$

and Equation (4.10) yields simply

$$R^3C' + \frac{f'}{3} + f\cot\Theta = \frac{\cot\Theta}{B}, \tag{4.12}$$

where we have utilised the identity (4.9) for $n = 1$. Equation (4.12) is in complete accord with the results given in the previous section, namely $F' = 0$ and $H = 0$, where F and H are defined by (3.8). In the remainder of this section we assume $n \neq 1, 2$.

Now on dividing (4.10) by $r^{2n-7}R^{n-3}$, we can group the terms not involving $\bar{\sigma}(\lambda)$ as R^6, R^3 or R^0, while the $\bar{\sigma}(\lambda)$ term behaves like r^{5-2n}/R^{n-1} which we can balance with the three groupings in any of the following ways:

$$r^6\frac{r^{-1-2n}}{R^{n-1}}, \qquad r^3\frac{r^{2-2n}}{R^{n-1}}, \qquad R^6\frac{r^{5-2n}}{R^{5+n}}, \qquad R^3\frac{r^{5-2n}}{R^{2+n}}, \qquad \frac{r^{5-2n}}{R^{n-1}},$$

which give rise to a multiple of λ only in the three special cases $n = 0$, 1 and 2, which are all excluded from the discussion. Accordingly, for $n \neq 0, 1, 2$ we need to assume that the term involving $\bar{\sigma}(\lambda)$ does not arise in Equation (4.10). This can be achieved either if

$$\frac{f'}{3} + f\cot\Theta = 0, \tag{4.13}$$

or if $\bar{\sigma}(\lambda) = \sigma_o/\lambda^2$, where σ_0 is a constant. Assuming that one of these apply, we may readily deduce the following equations from the terms involving R^6, R^3 and R^0, thus

$$\begin{aligned} &C' = 3(n-1)D,\\ &C\frac{f'}{3} + Df = 2(n-1)\frac{B^{-2/n}}{n}B',\\ &f\left\{\frac{Cf'}{3} + Df\right\} = -2(n-1)B^{1-2/n}\frac{f'}{3}, \end{aligned} \tag{4.14}$$

and the latter two equations are only consistent if

$$\frac{fB'}{n} = -\frac{Bf'}{3}, \tag{4.15}$$

which integrates to give $Bf^{n/3} =$ constant. Now on comparison of (4.9) and $(4.14)_2$ we obtain an equation which can be readily integrated to yield

$$B(\Theta) = B_0(\sin\Theta)^{n/(n-1)}, \tag{4.16}$$

where B_0 denotes an arbitrary constant. Thus, from the previous integral we have

$$f(\Theta) = f_0(\sin\Theta)^{-3/(n-1)}, \tag{4.17}$$

where f_0 denotes an additional constant. We observe that this expression coincides with the solution of (4.13) only if $n = 2$, so that for other values of n we must have $\bar{\sigma}(\lambda) = \sigma_0/\lambda^2$. However, from the definition of $B(\Theta)$ (namely Equations (3.2) and (4.4)) it is straightforward to deduce that there are no values on n for which (4.16) and (4.17) are compatible, not even $n = 2$ which excludes the option (4.13). Accordingly, for general n there are no values giving rise to non-trivial $f(\Theta)$. In the following section we present a detailed analysis of the special case $n = 2$.

5. Results for the special case of *n*=2

In this section we show that a non-trivial solution for $f(\Theta)$ exists for the special case $n = 2$, but we are not able to integrate fully the governing highly nonlinear ordinary differential equation. We need to assume that the $\bar{\sigma}(\lambda)$ term does not arise in the Equations (4.6) and (4.10). This is the case either if $\Sigma_0(\lambda)$ is a constant or if it is a constant times $\log\lambda$. In either case (4.10) becomes simply $C' = 3D$ while (4.6) and (4.9) yield, respectively,

$$Q_R = -R^2C - \frac{2R^2}{r^3}, \qquad Q_\Theta = C\frac{f'}{3} + Dr^3 + \frac{2}{3}\frac{f'}{r^3},$$

$$C\frac{f'}{3} + Df = \cot\Theta + \frac{B'}{2B}, \tag{5.1}$$

from which Q may be readily integrated to give

$$Q(R,\Theta) = -\frac{R^3C}{3} + \log\left(r^2\sin\Theta B(\Theta)^{1/2}\right) + Q_0, \tag{5.2}$$

where Q_0 denotes an arbitrary constant. Now from $C' = 3D$ and $(5.1)_3$ we can, by integration, obtain

$$Cf = 3\log(\sin\Theta B(\Theta)^{1/2}) + C_1, \tag{5.3}$$

where C_1 denotes the constant of integration. Thus, from $(4.7)_1$ and (5.3) we may deduce the highly nonlinear ordinary differential equation for $f(\Theta)$, namely

$$f\left\{\frac{(Bf')'}{3} + \cot\Theta\frac{Bf'}{3} - 6Bf\right\} = 3\log(\sin\Theta B^{1/2}) + C_1, \tag{5.4}$$

where in this case $B(\Theta)$ as a function of f and f' is defined by

$$B(\Theta) = \frac{1}{(f^2 + f'^2/9)}. \tag{5.5}$$

Equations (5.4) and (5.5) constitute a well-defined second-order ordinary differential equation for the determination of $f(\Theta)$. At least in principle, for this particular response function, we know there exists nontrivial $f(\Theta)$, perhaps involving three arbitrary constants. We can achieve some minor simplification of (5.4) and (5.5) by introducing ρ such that

$$f = \frac{\cos\rho}{B^{1/2}}, \qquad \frac{f'}{3} = \frac{\sin\rho}{B^{1/2}}, \tag{5.6}$$

in which case we have

$$\frac{B'}{2B} = -(\rho' + 3)\tan\rho, \tag{5.7}$$

while (5.4) becomes

$$(\sin\Theta\sin 2\rho)' = 3\sin\Theta\cos 2\rho + 6\sin\Theta\log(\sin\Theta B^{1/2}) + (2C_1 + 9)\sin\Theta. \tag{5.8}$$

However, such transformations appear not to be effective in terms of producing simple analytical solutions and further results can only be obtained numerically.

6. Application to the eversion of a spherical cap

The controllable deformation involved in turning a spherical shell inside out and due originally to Green and Shield [6] is given by

$$r = (-R^3 + K)^{1/3}, \qquad \theta = \pi - \Theta, \qquad \phi = \Phi, \tag{6.1}$$

where K is a constant. Namely, this deformation describes the eversion problem for a complete thick-walled spherical shell, which is everted by means of a cut. For other contributions to this problem, we refer the reader to Eringen [8, pp. 182–185]. For a portion of a spherical shell, a deformation of the form (1.3) is more likely to apply and the question arises as to whether we might utilise (1.4), for example, to describe the 'snap-buckling' of a spherical cap which is a familiar physical effect. For the majority of exact deformations which apply to particular finite elastic materials, it is not usually possible to satisfy stress boundary conditions in a point wise sense, and at best only approximate or 'averaged' stress conditions on the boundary can be satisfied. In many instances such solutions lead to useful practical load-deflection relations (see for example Klingbeil and Shield [9] and Hill and Lee [10]). We comment that in this section we have retained the usual convention in finite elasticity of using the capital letters A and B to designate the inner and outer radii. There should be no confusion with the previously introduced functions $A(\Theta)$ and $B(\Theta)$ which are not required in this section.

From Appendix A and Equation (A12) we have that the resultant force F^* acting in the z-direction, for the deformation given by (1.3) and (1.4), on a spherical cap of radius R and subtended by an angle Θ_0 is given by the expression

$$F^* = \pi R^2 \sin^2\Theta_0 \left\{\alpha + 2\beta\lambda - p_0\lambda^2\right\}, \tag{6.2}$$

where λ is defined by $(3.1)_2$ thus

$$\lambda = \frac{r}{R} = \left(\frac{1}{\xi_0^3} - 1\right)^{1/3}, \tag{6.3}$$

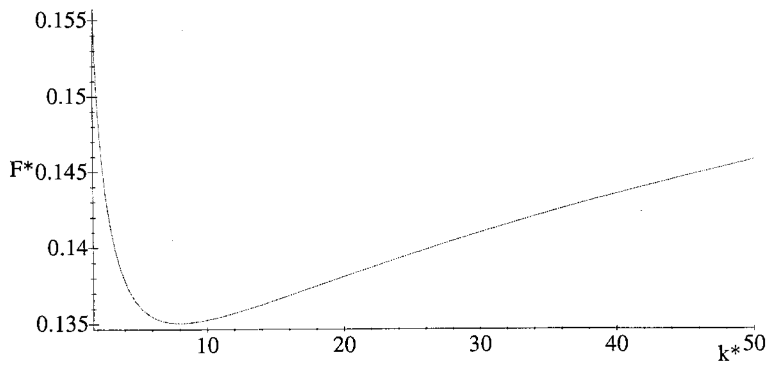

Figure 1. Variation of $F/\pi\alpha A^2$ with $k^* = k/A^3$ as given by (6.8) and (6.9) for $\Theta_0 = \pi/6$ and $B/A = 6/5$.

where ξ_0 is as defined in Appendix A, namely, Equation $(A10)_1$. We observe that α and β in (6.2) are the material constants arising in the modified Varga strain-energy function (1.2), and the limited experimental investigation given in [5] indicates that both α and β are positive and that β/α is approximately $1/15$. We also note that the resultant force F^* is quadratic in the principal stretch λ. Assuming for the time being that we may neglect the flat surface $\Theta = \Theta_0$ of the spherical cap which we take to be defined by

$$\{(R, \Theta, \Phi) : A \leqslant R \leqslant B; 0 \leqslant \Theta \leqslant \Theta_0; 0 \leqslant \Phi \leqslant 2\pi\},$$

where A and B here denote the inner and outer radii respectively, we can in principle determine the remaining unknown constants k and p_0 such that the condition F^* vanishing on the inner and outer spherical surfaces is satisfied, which would produce an approximate solution to the snap-buckling problem. This implies that the values of λ at the inner and outer surfaces, namely λ_A and λ_B are determined as roots of the quadratic equation

$$p_0\lambda^2 - 2\beta\lambda - \alpha = 0, \tag{6.4}$$

thus

$$\lambda_A = \frac{\beta + (\beta^2 + \alpha p_0)^{1/2}}{p_0}, \qquad \lambda_B = \frac{\beta - (\beta^2 + \alpha p_0)^{1/2}}{p_0}, \tag{6.5}$$

and therefore

$$\lambda_A + \lambda_B = \frac{2\beta}{p_0}, \qquad \lambda_A\lambda_B = -\frac{\alpha}{p_0}. \tag{6.6}$$

It is clear that for positive α, β, λ_A and λ_B these conditions are not compatible and therefore at least for this particular material, the phenomena of snap buckling of a spherical cap is not embodied in the deformation given by (1.3) and (1.4).

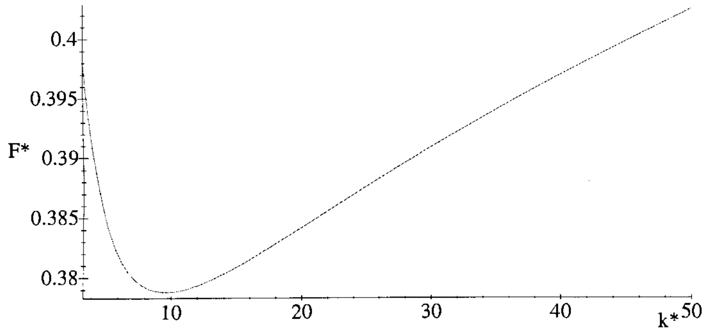

Figure 2. Variation of $F/\pi\alpha A^2$ with $k^* = k/A^3$ as given by (6.8) and (6.9) for$\Theta_0 = \pi/6$ and $B/A = 3/2$.

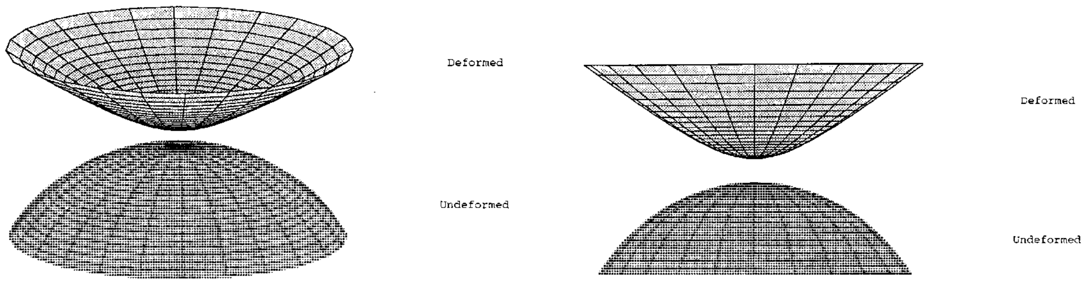

Figure 3. Actual deformation of the outer surface for $B/A = 3/2$ and $k^* = 8$.

Alternatively, we can suppose that the resultant force F^* vanishes on one of the spherical surfaces and is prescribed on the other, thus

$$F^*(A) = 0, \qquad F^*(B) = F, \tag{6.7}$$

in which case we may deduce

$$F = \frac{\pi B^2}{\lambda_A^2} \sin^2 \Theta_0 (\lambda_A - \lambda_B)\{\alpha(\lambda_A + \lambda_B) + 2\beta\lambda_A\lambda_B\}, \tag{6.8}$$

noting that $\lambda_A > \lambda_B$ since $\lambda(R)$ is monotonically decreasing. Figures 1 and 2 show the variation of $F/\pi\alpha A^2$ with $k^* = k/A^3$ for $\Theta_0 = \pi/6$ and for two values of B/A and assuming that $\beta/\alpha = 1/15$. It is clear from these figures that, as noted previously, F can never be zero and that for any given F there are two possible values of k. Further, in this context λ_A and λ_B are given by

$$\lambda_A = (k^*(\cos 2\Theta_0)^{-3/2} - 1)^{1/3}, \qquad \lambda_B = \left(\frac{k^*(\cos 2\Theta_0)^{-3/2}}{\delta^3} - 1\right)^{1/3}, \tag{6.9}$$

where δ denotes B/A and the actual deformation of the outer surface is shown in Figure 3 for the case $B/A = 3/2$ and $k^* = 8$.

We note that from Equation (B7) of Appendix B, as expected, the resultant force G acting in the conventional z-direction on the slanting surface given originally by $\Theta = \Theta_0$ becomes,

$$G^* = -\pi \sin^2 \Theta_0\{B^2(\alpha + 2\beta\lambda_B - p_0\lambda_B^2) - A^2(\alpha + 2\beta\lambda_A - p_0\lambda_A^2)\}, \tag{6.10}$$

and therefore in this instance $G^* = -F$ where F is given by (6.8). This, of course, is necessary in order that there are no net forces acting on the cap.

7. Concluding remarks

For the perfectly elastic incompressible modified Varga material, we have determined a new simple exact solution which corresponds to the eversion of a portion of a thick-walled spherical shell. We have demonstrated that for this particular strain-energy function, there does not exist a more general solution of this type. We have also investigated such deformations for a family of response functions which are characterised by an index n, and we have shown that other than the Varga material, only that for $n = 2$ admits a nontrivial deformation of the type examined here. We have attempted to utilize the new solution to solve the familiar problem of the snap-buckling of a portion of a spherical shell, but unfortunately our solution is not sufficiently general to accommodate the necessary stress boundary conditions, even when these are replaced by average force requirements. Instead we have used the solution for the problem of the eversion of a spherical shell, under no applied forces acting on one of the spherical surfaces.

Appendix A. Calculation for resultant force on the spherical surface of a cap

As described in Hill [7], the resultant force F^* acting in the conventional z-direction on a spherical cap of original radius R and subtended by an angle Θ_0, may be shown to be determined by the expression

$$F^* = 2\pi \int_0^{\Theta_0} [-pr \sin\theta (r \sin\theta)_\Theta + R^2 \sin\Theta \phi_1 (r\cos\theta)_R] \, d\Theta, \tag{A1}$$

where p is the pressure function arising in the equilibrium equations (2.3) and ϕ_1 is the first response function defined by $(2.5)_1$. Now for the axially symmetric deformation (1.3) for the modified Varga material, $f(\Theta)$ is given by (1.4), while the pressure function p is determined from (2.13) and we find that (A1) becomes

$$F^* = 2\pi \int_0^{\Theta_0} \left\{ \left[-p_0 + \frac{(\phi_1 + \beta)}{\lambda} \right] r (r \sin\Theta)_\Theta + \phi_1 \frac{R^4}{r^2} \cos\Theta \right\} \sin\Theta \, d\Theta. \tag{A2}$$

We choose to express this integral in the following manner,

$$F^* = F_1^* + F_2^* + F_3^*, \tag{A3}$$

where F_1^*, F_2^* and F_3^*, are taken to be

$$\begin{aligned} F_1^* &= -2\pi p_0 \int_0^{\Theta_0} r \sin\Theta (r \sin\Theta)_\Theta \, d\Theta = -\pi p_0 (-R^3 + f(\Theta_0))^{2/3} \sin^2\Theta_0, \\ F_2^* &= 2\pi R \int_0^{\Theta_0} \frac{\phi_1}{r^2} \{(-R^3 + f)\cos\Theta + f \sin\Theta \tan 2\Theta + R^3 \cos\Theta\} \sin\Theta \, d\Theta, \\ F_3^* &= 2\pi\beta R \int_0^{\Theta_0} \frac{1}{r^2} \{(-R^3 + f)\cos\Theta + f\sin\Theta\tan 2\Theta\} \sin\Theta \, d\Theta, \end{aligned} \tag{A4}$$

where we have utlilised (2.14). Now F_1^* is trivial and has been evaluated immediately, while F_2^* can be shown to become

$$F_2^* = \pi R \int_0^{\Theta_0} \frac{\phi_1}{r^2} f \tan 2\Theta \, d\Theta,$$

which on use of $(3.5)_2$ gives simply

$$F_2^* = \pi R \int_0^{\Theta_0} (\alpha R + \beta r) \sin 2\Theta \, d\Theta, \tag{A5}$$

where we have used $B(\Theta) = k^{-1}(\cos 2\Theta)^{5/2}$. Part of $(A5)$ integrates immediately, while for the remainder we make the substitution $u = \cos 2\Theta$ to obtain

$$F_2^* = \pi \alpha R^2 \sin^2 \Theta_0 + \frac{\pi \beta R}{2} \int_{u_0}^1 (-R^3 + ku^{-3/2})^{1/3} \, du, \tag{A6}$$

where u_0 denotes $\cos 2\Theta_0$. Now F_3^* simplifies to give

$$F_3^* = \pi \beta R \int_0^{\Theta_0} \left\{ -R^3 + \frac{f}{\cos 2\Theta} \right\} \frac{\sin 2\Theta}{r^2} \, d\Theta,$$

which becomes

$$F_3^* = -\frac{\pi \beta R^4}{2} \int_{u_0}^1 \frac{du}{(-R^3 + ku^{-3/2})^{2/3}} + \frac{\pi \beta R}{2} \int_{u_0}^1 \frac{ku^{-5/2} \, du}{(-R^3 + ku^{-3/2})^{2/3}},$$

and on making the substitution $v = ku^{-3/2}$ in the second integral, we can perform the integration to yield

$$\begin{aligned} F_3^* \ &= \ \pi \beta R \{(-R^3 + f(\Theta_0))^{1/3} - (-R^3 + k)^{1/3}\} \\ &\quad - \frac{\pi \beta R^4}{2} \int_{u_0}^1 \frac{du}{(-R^3 + ku^{-3/2})^{2/3}}. \end{aligned} \tag{A7}$$

Thus altogether we find that F^* becomes

$$\begin{aligned} \frac{F^*}{\pi} \ &= \ -p_0 r_0^2 \sin^2 \Theta_0 + \alpha R^2 \sin^2 \Theta_0 + \beta R r_0 - \beta R (-R^3 + k)^{1/3} \\ &\quad - \beta R^4 \int_{u_0}^1 \frac{du}{(-R^3 + ku^{-3/2})^{2/3}} + \frac{\beta R}{2} \int_{u_0}^1 \frac{ku^{-3/2} du}{(-R^3 + ku^{-3/2})^{2/3}}, \end{aligned} \tag{A8}$$

where r_0 denotes $(-R^3 + f(\Theta_0))^{1/3}$. On making the substitution $\xi = Ru^{1/2}/k^{1/3}$ we may eventually deduce

$$\begin{aligned} \frac{F^*}{\pi} \ &= \ -p_0 r_0^2 \sin^2 \Theta_0 + \alpha R^2 \sin^2 \Theta_0 + \beta R r_0 - \beta R (-R^3 + k)^{1/3} \\ &\quad + \beta k^{2/3} \int_{\xi_0}^{R/k^{1/3}} \frac{(1 - 2\xi^3) d\xi}{(1 - \xi^3)^{2/3}}, \end{aligned} \tag{A9}$$

where ξ_0 denotes $Ru_0^{1/2}/k^{1/3}$, namely $R(\cos 2\Theta_0)^{1/2}/k^{1/3}$. Thus with the notation

$$\xi_0 = Ru_0^{1/2}/k^{1/3}, \qquad \xi_1 = R/k^{1/3}, \tag{A10}$$

Equation (A9) simplifies to yield

$$\frac{F^*}{\pi R^2} = \frac{(1-u_0)}{2}\left\{\alpha - \frac{p_0}{\xi_0^2}(1-\xi_0^3)^{2/3}\right\} + \beta\left\{\frac{(1-\xi_0^3)^{1/3}}{\xi_0} - \frac{(1-\xi_1^3)^{1/3}}{\xi_1}\right\}$$
$$+\frac{\beta}{\xi_1^2}\int_{\xi_0}^{\xi_1}\frac{(1-2\xi^3)\mathrm{d}\xi}{(1-\xi^3)^{2/3}}. \tag{A11}$$

For the integral in (A11) we have

$$\int_{\xi_0}^{\xi_1}\left\{(1-\xi^3)^{1/3} - \frac{\xi^3}{(1-\xi^3)^{2/3}}\right\}\mathrm{d}\xi = \int_{\xi_0}^{\xi_1}\frac{\mathrm{d}}{\mathrm{d}\xi}\{\xi(1-\xi^3)^{1/3}\}\mathrm{d}\xi,$$

which on integration, yields altogether

$$\frac{F^*}{\pi R^2} = (1-u_0)\left\{\frac{\alpha}{2} + \beta\frac{(1-\xi_0^3)^{1/3}}{\xi_0} - p_0\frac{(1-\xi_0^3)^{2/3}}{2\xi_0^2}\right\}, \tag{A12}$$

and this is the required expression for the resultant force F^* acting in the z-direction on a spherical cap of radius R and subtended by an angle Θ_0.

Appendix B. Calculation for resultant force on the slanting surface of a cap

The resultant force G^* acting in the conventional z-direction on the slanting surface of a spherical cap, which is subtended by an angle Θ, is calculated as follows. In terms of the Cauchy stress vector t^j and the first Piola–Kirchoff stress vector t_R^j we have

$$\mathrm{d}G^* = \left(t^1\cos\theta - rt^2\sin\theta\right)\mathrm{d}a = (t_R^1\cos\theta - rt_R^2\sin\theta)\mathrm{d}A, \tag{B1}$$

where $\mathrm{d}A$ and $\mathrm{d}a$ represent elementary undeformed and deformed areas and in particular $\mathrm{d}A$ is given by

$$\mathrm{d}A = R\sin\Theta\,\mathrm{d}R\,\mathrm{d}\Phi. \tag{B2}$$

Now the slanting surface subtended by an angle Θ has normal vector $\boldsymbol{n}_R = (0, R, 0)$ and in terms of the Cauchy stress tensor t^{ij} and the first Piola–Kirchoff stress tensor t_R^{Kj} we have

$$t_R^j = t_R^{Kj}n_{RK} = X_{,i}^K t^{ij} n_{RK}, \tag{B3}$$

so that on making use of

$$\Theta_r = -\frac{r^2\sin\theta}{R^2\sin\Theta}\theta_R, \quad \Theta_\theta = \frac{r^2\sin\theta}{R^2\sin\Theta}r_R, \tag{B4}$$

and the expressions given in Hill [7] for the Cauchy stress tensor, we may deduce from the above equations

$$G^* = 2\pi \int_A^B [pr \sin\theta (r\sin\theta)_R + \sin\Theta \phi_1 (r\cos\theta)_\Theta] \mathrm{d}R, \tag{B5}$$

noting that the Φ integration can be performed directly.

In particular, for the deformation given by (1.3) and (1.4) for the modified Varga material, the pressure function p is given by (2.13) where ϕ_1 is determined from $(3.5)_2$ and from (B5) we may obtain

$$G^* = -2\pi \sin^2\Theta \int_A^B \left[\alpha R + \beta (rR)_R + p_0 \frac{R^2}{r}\right] \mathrm{d}R, \tag{B6}$$

which can be readily integrated to yield

$$G^* = -\pi \sin^2\Theta \{\alpha(B^2 - A^2) + 2\beta(Br_B - Ar_A) - p_0(r_B^2 - r_A^2)\}, \tag{B7}$$

where r_A and r_B denote the values of r evaluated at $R = A$ and $R = B$, respectively. From this equation, for $\Theta = \Theta_0$, we can readily deduce (6.10) where λ_A and λ_B are defined by (6.9).

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